

# DEGENERATING HODGE STRUCTURE OF ONE-PARAMETER FAMILY OF CALABI–YAU THREEFOLDS

TATSUKI HAYAMA    ATSUSHI KANAZAWA

**ABSTRACT.** To a one-parameter family of Calabi–Yau threefolds, we can associate the extended period map by the log Hodge theory of Kato and Usui. In the present paper, we study the image of a maximally unipotent monodromy point under the extended period map. As an application, we prove the generic Torelli theorem for a large class of one-parameter families of Calabi–Yau threefolds.

## 1. INTRODUCTION

The present paper is concerned with the limit mixed Hodge structure around a maximally unipotent monodromy (MUM) point of a one-parameter family of Calabi–Yau threefolds whose Kodaira spencer map is generically an isomorphism. For such a family, the period domain for the Hodge structures and the limit mixed Hodge structures (LMHSs) were previously studied by [KU] and [GGK]. The starting point of the present work is the theory of normalization of the LMHS around a MUM point developed in [GGK]. MUM points play a central role in mirror symmetry [CdOGP, Mor]. Mirror symmetry is a duality between complex geometry and symplectic geometry among several Calabi–Yau threefolds. It expects that each MUM point of a family of Calabi–Yau threefolds corresponds to a *mirror* Calabi–Yau threefold of the family. For a large class of Calabi–Yau threefolds, we observe that the normalization of the LMHS reflects the topological invariants of mirror Calabi–Yau threefold.

The idea of this paper is to investigate the degenerating Hodge structures in the framework of the log Hodge theory [KU]. An advantage of our approach is that, by slightly extending the domain and range of the period map, we have a better control of the period map. As an application, we prove the generic Torelli theorem for a large class of one-parameter families of Calabi–Yau threefolds (Theorem 4.3). The generic Torelli theorem was confirmed for the mirror families of Calabi–Yau hypersurfaces in weighted projective spaces by Usui [Usu2] and Shirakawa [Shi]. Our study is a slight refinement of their technique but can be applied to a wider class of Calabi–Yau threefolds. The result is particularly interesting when a family has multiple MUM points and also works for new examples beyond toric geometry such as the mirror family of the Pfaffian–Grassmann Calabi–Yau threefolds (Section 5.2).

The layout of this paper is as follows. Section 2 covers some basics of Hodge theory and the compactification of period domains. This chapter also serves to set notations. Section 3 begins with a review of the normalization of a LMHS obtained in [GGK]. We then study the LMHSs using the normalization. Section 4 is devoted to the generic

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2010 *Mathematics Subject Classification.* 14C30, 14C34, 14J32.

*Key words and phrases.* (log) Hodge theory, Calabi–Yau, Torelli problem, mirror symmetry.

Torelli for a one-parameter family of Calabi–Yau threefolds. Section 5 briefly reviews mirror symmetry of Calabi–Yau threefolds with a particular emphasis on a monodromy transformation around a MUM point. We also discuss some suggestive examples of Calabi–Yau threefolds with two MUM points.

**Acknowledgement.** It is a pleasure to record our thanks to C. Nakayama for useful comments on the preliminary version of the present paper. This research was partially supported by Research Fund for International Young Scientists NSFC 11350110209 (Hayama).

## 2. LMHS AND PARTIAL COMPACTIFICATION OF PERIOD DOMAIN

**2.1. Hodge structure and period domain.** In this section, we recall the definition of polarized Hodge structures and of period domains. A Hodge structure of weight  $w$  with Hodge numbers  $(h^{p,q})_{p,q}$  is a pair  $(H, F)$  consisting of a free  $\mathbb{Z}$ -module  $H$  of rank  $\sum_{p,q} h^{p,q}$  and a decreasing filtration  $F$  on  $H_{\mathbb{C}} := H \otimes \mathbb{C}$  satisfying the following conditions:

- (1)  $\dim_{\mathbb{C}} F^p = \sum_{r \geq p} h^{r, w-r}$  for all  $p$ ;
- (2)  $H_{\mathbb{C}} = \bigoplus_{p+q=w} H^{p,q}$  ( $H^{p,q} := F^p \cap \overline{F^{w-p}}$ ).

For Hodge structures  $(H, F)$  and  $(H', F')$ , homomorphism  $f : H \rightarrow H'$  is a  $(r, r)$ -morphism of Hodge structures if  $f(F^p) \subset F'^{p+r}$  and  $f(\bar{F}^p) \subset \bar{F}'^{p+r}$ .

A polarization  $\langle *, ** \rangle$  for a Hodge structure  $(H, F)$  of weight  $w$  is a non-degenerate bilinear form on  $H$ , symmetric if  $w$  is even and skew-symmetric if  $w$  is odd, satisfying the following conditions:

- (3)  $\langle F^p, F^q \rangle = 0$  for  $p + q > w$ ;
- (4)  $i^{p-q} \langle v, \bar{v} \rangle > 0$  for  $0 \neq v \in H^{p,q}$ .

We fix a polarized Hodge structure  $(H_0, F_0, \langle *, ** \rangle_0)$  of weight  $w$  with Hodge numbers  $(h^{p,q})_{p,q}$ . We define the period domain  $D$  which parametrizes all Hodge structures of this type by

$$D := \left\{ F \mid \begin{array}{l} (H_0, F, \langle *, ** \rangle_0) \text{ is a polarized Hodge structure} \\ \text{of weight } w \text{ with Hodge numbers } (h^{p,q})_{p,q} \end{array} \right\}.$$

The compact dual  $\check{D}$  of  $D$  is

$$\check{D} := \{ F \mid (H_0, F, \langle *, ** \rangle_0) \text{ satisfies the above (1)–(3)} \}.$$

Let  $G_A := \text{Aut}(H_0 \otimes A, \langle *, ** \rangle_0)$  for a  $\mathbb{Z}$ -module  $A$ . Then,  $G_{\mathbb{R}}$  acts transitively on  $D$  and  $G_{\mathbb{C}}$  acts transitively on  $\check{D}$ .

Let  $S$  be a complex manifold. A variation of Hodge structure (VHS) over  $S$  is a pair  $(\mathcal{H}, \mathcal{F})$  consisting of a  $\mathbb{Z}$ -local system and a filtration of  $\mathcal{H} \otimes \mathcal{O}_S$  over  $S$  satisfying the following conditions:

- (1) The fiber  $(H_s, F_s)$  at  $s \in S$  is a Hodge structure;
- (2)  $\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega_S^1$  for the connection  $\nabla := id \otimes d : \mathcal{H} \otimes \mathcal{O}_S \rightarrow \mathcal{H} \otimes \Omega_S^1$ .

A polarization for a VHS is a bilinear form on the local system which defines a polarization on each fiber. In this paper, a VHS is always assumed to be polarized.

For a VHS over  $S$ , we fix a base point  $s_0 \in S$ . Let  $D$  be the period domain for the Hodge structure at  $s_0$ . We then have the period map  $\phi : S \rightarrow \Gamma \backslash D$  via  $s \mapsto F_s$ , where  $\Gamma$  is the monodromy group.

**2.2. Limit mixed Hodge structure.** Let  $\bar{S}$  be a smooth compactification of  $S$  such that  $\bar{S} - S$  is a normal crossing divisor. For each  $p \in \bar{S} - S$ , there exists a neighbourhood  $V$  of around  $p$  in  $\bar{S}$  such that  $U := V \cap S \cong (\Delta^*)^m \times \Delta^{n-m}$  where  $\Delta$  is the unit disk. We can lift the period map to  $\tilde{\phi} : \tilde{U} \rightarrow D$ , where  $\tilde{U} \rightarrow U$  is the universal covering map. Under the identification  $\tilde{U} \cong \mathcal{H}^m \times \Delta^{n-m}$ , the covering map  $\tilde{U} \rightarrow U$  is given by

$$(z_1, \dots, z_n) \mapsto (\exp(2\pi i z_1), \dots, \exp(2\pi i z_m), z_{m+1}, \dots, z_n).$$

Let  $T_1, \dots, T_m$  be a generator of the monodromy around  $p$  such that

$$\tilde{\phi}(\dots, z_j + 1, \dots) = T_j \tilde{\phi}(\dots, z_j, \dots).$$

Let us assume  $T_j$  is unipotent. Then  $N_j = \log T_j$  is nilpotent in the Lie algebra  $\mathfrak{g}_{\mathbb{Q}}$ , and  $N_1, \dots, N_m$  are commuting with each other. We define  $\tilde{\psi} : \tilde{U} \rightarrow \check{D}$  by  $z \mapsto \exp(-\sum_j z_j N_j) \phi(z)$ . Since  $\tilde{\psi}(\dots, z_j + 1, \dots) = \psi(\dots, z_j, \dots)$ ,  $\tilde{\psi}$  descends to  $\psi : U \rightarrow \check{D}$ , which admits a unique extension to  $\psi : \Delta^n \rightarrow \check{D}$  by [Sch]. We call  $F_{\infty} := \psi(0) \in \check{D}$  the limit Hodge filtration (LHF).

**Remark 2.1.** The LHF is not uniquely determined by a VHS. In fact, for  $f_j \in \mathcal{O}_{\Delta}$ , we obtain new coordinates

$$(\exp(2\pi i f_1(z_1))z_1, \dots, \exp(2\pi i f_n(z_n))z_n),$$

with respect to which, the LHF is given by  $\exp(-\sum f_j(0)N_j)F_{\infty}$ . Moreover,  $N_1, \dots, N_m$  depend also on the choice of coordinates. However the nilpotent orbit (to be discussed in the next subsection) is determined by the VHS.

Let  $N := N_1 + \dots + N_m$ . By [Sch], we have an increasing filtration  $W(N)$  of  $H_{\mathbb{R},0} := H_0 \otimes \mathbb{R}$ . Denoting by  $W$  the shifted filtration of  $W(N)$  by the weight  $w$ , the pair  $(W, F_{\infty})$  has the following properties:

- (1) the graded quotient  $(\text{Gr}_k^W, F_{\infty} \text{Gr}_{k,\mathbb{C}}^W)$  is a Hodge structure of weight  $k$ ;
- (2)  $N$  defines a  $(-1, -1)$ -morphism  $(\text{Gr}_k^W, F_{\infty} \text{Gr}_{k,\mathbb{C}}^W) \rightarrow (\text{Gr}_{k-2}^W, F_{\infty} \text{Gr}_{k-2,\mathbb{C}}^W)$  of Hodge structures;
- (3)  $N^k : (\text{Gr}_{w+k}^W, F_{\infty} \text{Gr}_{w+k,\mathbb{C}}^W) \rightarrow (\text{Gr}_{w-k}^W, F_{\infty} \text{Gr}_{w-k,\mathbb{C}}^W)$  is isomorphism;
- (4)  $\langle *, N^k(**) \rangle$  gives a polarization on  $(\text{Gr}_{w+k}^W, F_{\infty} \text{Gr}_{w+k,\mathbb{C}}^W)$ .

The pair  $(W, F_{\infty})$  is called the limit mixed Hodge structure (LMHS).

**2.3. Partial compactification of period domain.** We call  $\sigma \subset \mathfrak{g}_{\mathbb{R}}$  a nilpotent cone if it satisfies the following conditions:

- (1)  $\sigma$  is a closed cone generated by finitely many elements of  $\mathfrak{g}_{\mathbb{Q}}$ ;
- (2)  $N \in \sigma$  is a nilpotent as an endomorphism of  $H_{\mathbb{R}}$ ;
- (3)  $NN' = N'N$  for any  $N, N' \in \sigma$ .

For  $A = \mathbb{R}, \mathbb{C}$ , we denote by  $\sigma_A$  the  $A$ -linear span of  $\sigma$  in  $\mathfrak{g}_A$ .

**Definition 2.2.** Let  $\sigma = \sum_{j=1}^n \mathbb{R}_{\geq 0} N_j$  be a nilpotent cone and  $F \in \check{D}$ . Then the pair consisting of  $\sigma$  and  $\exp(\sigma_{\mathbb{C}})F \subset \check{D}$  is called a nilpotent orbit if it satisfies the following conditions:

- (1)  $\exp(\sum_j i y_j N_j)F \in D$  for all  $y_j \gg 0$ .
- (2)  $NF^p \subset F^{p-1}$  for all  $p \in \mathbb{Z}$  and for all  $N \in \sigma$ .

The data  $(N_1, \dots, N_m, F_{\infty})$  given in the previous section generates a nilpotent orbit. Moreover, any nilpotent orbit generates a LMHS. In fact,  $W(N) = W(N')$  for any  $N$  and  $N'$  in the relative interior of  $\sigma$  (see [CK] for example), and the pair  $(W(N)[w], F')$  is a LMHS for any  $F' \in \exp(\sigma_{\mathbb{C}})F$ .

Let  $\Sigma$  be a fan consisting of nilpotent cones. We define the set of nilpotent orbits

$$D_{\Sigma} := \{(\sigma, Z) \mid \sigma \in \Sigma, (\sigma, Z) \text{ is a nilpotent orbit}\}.$$

For a nilpotent cone  $\sigma$ , the set of faces of  $\sigma$  is a fan, and we abbreviate  $D_{\{\text{faces of } \sigma\}}$  as  $D_{\sigma}$ . Let  $\Gamma$  be a subgroup of  $G_{\mathbb{Z}}$  and  $\Sigma$  a fan of nilpotent cones. We say  $\Gamma$  is compatible with  $\Sigma$  if  $\text{Ad}(\gamma)(\sigma) \in \Sigma$  for all  $\gamma \in \Gamma$  and for all  $\sigma \in \Sigma$ . Then  $\Gamma$  acts on  $D_{\Sigma}$  if  $\Gamma$  is compatible with  $\Sigma$ . Moreover we say  $\Gamma$  is strongly compatible with  $\Sigma$  if it is compatible with  $\Sigma$  and for all  $\sigma \in \Sigma$  there exists  $\gamma_1, \dots, \gamma_n \in \Gamma(\sigma) := \Gamma \cap \exp(\sigma)$  such that  $\sigma = \sum_j \mathbb{R}_{\geq 0} \log(\gamma_j)$ .

We consider the geometric structure of  $\Gamma(\sigma)^{\text{gp}} \backslash D_{\sigma}$  in the case where  $\sigma$  has rank 1 (we will discuss this case in the next section). For a nilpotent cone  $\sigma = \mathbb{R}_{\geq 0} N$  and the  $\mathbb{Z}$ -subgroup  $\Gamma(\sigma)^{\text{gp}} = e^{\mathbb{Z}N}$ , we have the partial compactification  $\Gamma(\sigma)^{\text{gp}} \backslash D_{\sigma}$ . We now show its geometric structure following the exposition of [KU]. Let us define

$$\mathbb{C} \times \check{D} \supset E_{\sigma} := \left\{ (s, F) \left| \begin{array}{l} \exp(\ell(s)N)F \in D \text{ if } s \neq 0, \\ (\sigma, \exp(\mathbb{C}N)F) \text{ is a nilpotent orbit if } s = 0 \end{array} \right. \right\},$$

where  $\ell(s)$  is a branch of  $\log(s)/2\pi i$ . Here  $\mathbb{C}$  is endowed with a log structure as a toric variety and  $\mathbb{C} \times \check{D}$  is a logarithmic analytic space. By [KU, Theorem A], the subspace  $E_{\sigma}$  is a log manifold with the map

$$E_{\sigma} \rightarrow \Gamma(\sigma)^{\text{gp}} \backslash D_{\sigma}, \quad (s, F) \mapsto \begin{cases} \exp(\ell(s)N)F & \text{if } s \neq 0, \\ (\sigma, \exp(\sigma_{\mathbb{C}})F) & \text{if } s = 0. \end{cases}$$

The geometric structure of  $\Gamma(\sigma)^{\text{gp}} \backslash D_{\sigma}$  is induced by the map above, which is a  $\mathbb{C}$ -torsor, i.e.  $\Gamma(\sigma)^{\text{gp}} \backslash D_{\sigma} \cong E_{\sigma}/\mathbb{C}$ .

**Theorem 2.3** ([KU, Theorem A]). *Let  $\Sigma$  be a fan of nilpotent cones and let  $\Gamma$  be a subgroup of  $G_{\mathbb{Z}}$  which is strongly compatible with  $\Sigma$ . Then the following hold:*

- (1) *If  $\Gamma$  is neat (i.e., the subgroup of  $\mathbb{G}_m(\mathbb{C})$  generated by all the eigenvalues of all  $\gamma \in \Gamma$  is torsion free), then  $\Gamma \backslash D_{\Sigma}$  is a logarithmic manifold.*
- (2) *The map  $\Gamma(\sigma)^{\text{gp}} \backslash D_{\sigma} \rightarrow \Gamma \backslash D_{\Sigma}$  is open and locally an isomorphism of logarithmic manifolds.*

Logarithmic manifolds are a generalization of analytic spaces introduced in [KU]. A logarithmic manifold is a subspace of a logarithmic analytic space, whose topology is induced by the strong topology.

For a VHS, locally the period map  $U \rightarrow \Gamma \backslash D$  can be extended to the map  $V \rightarrow \Gamma \backslash D_{\sigma}$ . We assume that there exists a fan  $\Sigma$  which includes all nilpotent cones arises from all

local monodromies arising from  $\bar{S} - S$ . Note that a construction of fans is still an open problem in higher dimensional case (cf. [Usu1, §4]). Then we have an extended period map  $\bar{S} \rightarrow \Gamma \backslash D_\Sigma$ . Although the target space is not an analytic space, we have the following result:

**Theorem 2.4** ([Usu1, §5]). *The image of  $\bar{S}$  is a compact analytic space if  $\bar{S}$  is compact.*

Moreover, the map is also analytic since the category of logarithmic analytic spaces is a full subcategory of  $\mathcal{B}(\log)$  whose objects are logarithmic manifolds ([KU, §3]).

### 3. THE CASE WHERE $\text{rank } H = 4$ WITH $h^{3,0} = h^{2,1} = 1$

In this section, we consider Hodge structures with Hodge numbers

$$h^{p,q} = 1 \text{ if } p+q=3, p, q \geq 0, \text{ and } h^{p,q} = 0 \text{ otherwise.}$$

In this case, the partial compactifications of the period domain  $D$  are well-studied in [KU, §12.3] and [GGK]. We see that  $\text{rank } H = 4$  and  $G_\mathbb{Z} = Sp(2, \mathbb{Z})$ . The period domain  $D$  is the flag domain  $Sp(2, \mathbb{R})/(U(1) \times U(1))$  of dimension 4. If  $\sigma$  generates a nilpotent orbit, then  $\sigma = \mathbb{R}_{\geq 0}N$  and  $N$  is one of the following types:

- (1)  $N^2 = 0$  and  $\dim \text{Im } N = 1$ ;
- (2)  $N^2 = 0$  and  $\dim \text{Im } N = 2$ ;
- (3)  $N^3 \neq 0$  and  $N^4 = 0$ .

The case (3) is called maximally unipotent monodromy (MUM). The goal of this section is to analyze MUM and their LMHS in detail.

**3.1. Normalization of monodromy matrix.** Let  $T \in G_\mathbb{Z}$  be a unipotent element such that  $\log T = N$  is a MUM element. The monodromy weight filtration  $W = W(N)[3]$  is

$$\{0\} = W_{-1} \subset W_0 = W_1 \subset W_2 = W_3 \subset W_4 = W_5 \subset W_6 = H_\mathbb{Q}$$

with the graded quotient  $\text{Gr}_{2p}^W \cong \mathbb{Q}$  for  $0 \leq p \leq 3$ . The pair  $(\text{Gr}_{2p}^W, F\text{Gr}_{2p, \mathbb{C}}^W)$  is the Tate Hodge structure of weight  $2p$  if  $(N, F)$  generates a nilpotent orbit. The LMHS condition induces

$$\text{Gr}_6^W \xrightarrow{N} \text{Gr}_4^W \xrightarrow{N} \text{Gr}_2^W \xrightarrow{N} \text{Gr}_0^W,$$

where each  $N : \text{Gr}_{2p}^W \rightarrow \text{Gr}_{2p-2}^W$  is an isomorphism of Hodge structures.

By [GGK, Lemma (I.B.1) & (I.B.3)], we may choose a symplectic basis  $e_0, \dots, e_3$  of  $H_\mathbb{Z}$  which satisfies

$$(3.1) \quad W_{2p} = \text{span}_\mathbb{R}\{e_j \mid 0 \leq j \leq p\} \quad (0 \leq p \leq 3), \quad (\langle e_i, e_j \rangle)_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

By [GGK, (I.B.7)], with respect to this basis,  $N$  is of the form

$$(3.2) \quad N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ e & b & 0 & 0 \\ f & e & -a & 0 \end{bmatrix}.$$

for some  $a, b, e, f \in \mathbb{Q}$ . The polarization condition of a LMHS yields inequalities:

$$i^6 \langle e_3, N^3 e_3 \rangle = a^2 b > 0, \quad i^4 \langle e_2, N e_2 \rangle = b > 0.$$

Moreover, we have

$$T = e^N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ e + \frac{ab}{2} & b & 1 & 0 \\ f - \frac{a^2 b}{6} & e - \frac{ab}{2} & -a & 1 \end{bmatrix} \in G_{\mathbb{Z}},$$

which shows that

$$(3.3) \quad a, b, e \pm \frac{ab}{2}, f - \frac{a^2 b}{6} \in \mathbb{Z}.$$

The symplectic basis  $e_3, \dots, e_0$  with the properties (3.1) is not unique; for any  $A \in G_{\mathbb{Z}}(W) := \text{Aut}(H, \langle *, * \rangle, W)$ , the new basis  $Ae_3, \dots, Ae_0$  will do. Any  $A \in G_{\mathbb{Z}}(W)$  is represented by a lower triangular matrix with 1's on the diagonal, and thus written as  $A = e^M$  with

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ p & 0 & 0 & 0 \\ r & q & 0 & 0 \\ s & r & -p & 0 \end{bmatrix}$$

where  $p, q, r, s$  satisfy the same condition as  $a, b, e, f$  in (3.3). Under the transformation  $N \rightarrow \text{Ad}(A)N$ , the entries  $a, b, e, f$  change as follows:

$$(3.4) \quad \begin{aligned} a &\mapsto a, & b &\mapsto b, & e &\mapsto e - bp + aq, \\ f &\mapsto f - 2ep + bp^2 - apq + 2ar. \end{aligned}$$

**Proposition 3.1** ([GGK, Proposition I.B.10]). *Under the action of  $G_{\mathbb{Z}}(W)$ ,  $b$  is invariant, and  $a$  is invariant up to  $\pm 1$ . Moreover, for  $m = \gcd(a, b)$ ,  $[e] \in \mathbb{Z}/m\mathbb{Z}$  is invariant if  $ab$  is even, and  $[2e] \in \mathbb{Z}/2m\mathbb{Z}$  is invariant if  $ab$  is odd.*

**3.2. Period map around boundary point.** Let  $(\mathcal{H}, \mathcal{F})$  be a VHS over  $\Delta^*$  with monodromy  $N$  of the form (3.2). Hereafter, we fix such a presentation with  $a, b, e, f$ . For the monodromy group  $\Gamma = \langle T \rangle$ , we have the period map  $\phi : \Delta^* \rightarrow \Gamma \backslash D$  and its lifting  $\tilde{\phi} : \mathcal{H} \rightarrow D$ . Now the new map  $\exp(-zN)\tilde{\phi}(z)$  descends to a holomorphic map over  $\Delta$ , we denote it by  $\psi(s)$  where  $s = \exp(2\pi iz)$ . Here  $F_{\infty} = \psi(0)$  is the LHF and then  $F_{\infty}^3 \cap \overline{F_{\infty}^3} \pmod{W_5}$  is generated by  $e_3$ . We may choose a generator

$$e_3 + \pi_2 e_2 + \pi_1 e_1 + \pi_0 e_0$$

of the subspace  $F_{\infty}^3$  for some  $\pi_2, \pi_1, \pi_0 \in \mathbb{C}$ . Then the subspace  $F_{\psi(s)}^3$  corresponding to  $\psi(s) \in \check{D}$  is generated by

$$\psi_3(s)e_3 + \psi_2(s)e_2 + \psi_1(s)e_1 + \psi_0(s)e_0$$

where  $\psi_i$  for  $0 \leq i \leq 3$  are some holomorphic functions on  $\Delta$  with  $\psi_3(0) = 1$  and  $\psi_i(0) = \pi_i$  for  $0 \leq i \leq 2$ . By untwisting  $\psi$ , a local frame of the subspace  $\mathcal{F}^3$  spanned

by the period is given by

$$\begin{bmatrix} \omega_3(s) \\ \omega_2(s) \\ \omega_1(s) \\ \omega_0(s) \end{bmatrix} := \exp(zN) \begin{bmatrix} \psi_3(s) \\ \psi_2(s) \\ \psi_1(s) \\ \psi_0(s) \end{bmatrix}.$$

Here  $\omega_3(s) = \psi_3(s)$  and

$$\omega_2(s) = a\omega_3(s) \frac{\log(s)}{2\pi i} + \psi_2(s).$$

Therefore

$$(3.5) \quad q(s) := \exp\left(2\pi i \frac{\omega_2(s)}{a\omega_3(s)}\right) = \exp\left(2\pi i \frac{\psi_2(s)}{a\psi_3(s)}\right)s$$

defines a new coordinate of  $\Delta$ , which is known as the *mirror map* in mirror symmetry.

By §2.3, we have the extended period map  $\phi : \Delta \rightarrow \Gamma \backslash D_\sigma$ . As we saw in §2.3, the geometric structure of the image  $\phi(\Delta) \subset \Gamma \backslash D_\sigma$  is induced by the  $\mathbb{C}$ -torsor  $E_\sigma \rightarrow \Gamma(\sigma)^{\text{gp}} \backslash D_\sigma$ .

**Lemma 3.2.** *The period map  $\phi : \Delta \rightarrow \phi(\Delta)$  is an isomorphism as analytic spaces.*

*Proof.* The coordinate  $q$  gives a local section of the  $\mathbb{C}$ -torsor  $E_\sigma \rightarrow \Gamma(\sigma)^{\text{gp}} \backslash D_\sigma$  restricted on the image  $\phi(\Delta)$ . In fact, we can define

$$\rho : \phi(\Delta) \rightarrow E_\sigma; \quad \phi(s) \mapsto (q(s), \exp\left(-\frac{\psi_2(s)}{a\psi_3(s)}N\right)\psi(s)).$$

This induces isomorphisms  $\Delta \cong \rho(\phi(\Delta)) \cong \phi(\Delta)$  as analytic spaces.  $\square$

Moreover the map  $\Delta \rightarrow \phi(\Delta)$  induces an isomorphism of log structures in a manner similar to [Usu2, §4–5].

**3.3. Normalization of LHF.** Let  $(\sigma, \exp(\sigma_{\mathbb{C}})F)$  be a nilpotent orbit, i.e.  $(W, F)$  is a LMHS. We show that we have a canonical choice of  $F$  which has a normalized form with respect to the symplectic basis  $e_3, \dots, e_0$ .

For the LMHS  $(W, F)$ , we have the Deligne decomposition  $H_{\mathbb{C}} = \bigoplus_{0 \leq j \leq 3} I^{j,j}$  so that

$$W_{2p} = \bigoplus_{k \leq p} I^{k,k}, \quad F^p = \bigoplus_{k \geq p} I^{k,k}$$

for  $0 \leq p \leq 3$ . We can take a unique generator  $v_p \in I^{p,p}$  such that  $[v_p] = [e_p]$  in  $\text{Gr}_{2p, \mathbb{C}}^W$ . By [GGK, Proposition (I.C.2)], with respect to the basis  $v_3, \dots, v_0$ , the matrix  $N$  is in the form

$$N_\omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -a & 0 \end{bmatrix}.$$

Moreover, by [GGK, Proposition (I.C.4)], the period matrix of  $F$  is then written as

$$(3.6) \quad \begin{bmatrix} \omega_3 & \omega_2 & \omega_1 & \omega_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \pi_2 & 1 & 0 & 0 \\ \pi_1 & \frac{b}{a}\pi_2 + \frac{e}{a} & 1 & 0 \\ \pi_0 & \frac{e}{a}\pi_2 + \frac{f}{a} - \pi_1 & -\pi_2 & 1 \end{bmatrix}.$$

By multiplying  $\exp(-\frac{\pi_2}{a}N)$ , we may further choose  $F$  so that  $\pi_2 = 0$ . If  $\pi_2 = 0$ , by the second bilinear relation [GGK, (I.C.10)], the period matrix (3.6) is written as

$$(3.7) \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f/2a & e/a & 1 & 0 \\ \pi & f/2a & 0 & 1 \end{bmatrix}.$$

Here the values  $f/2a$ ,  $e/a$  and  $\pi$  correspond to the extension class of the LMHS [GGK, §I.C]. We observe that the boundary component  $D_\sigma \setminus D \cong \mathbb{C}$  is parametrized by  $\pi$ .

Recall that the LHF depends on the choice of coordinates for a VHS (Remark 2.1). If we use the canonical coordinate  $q$  of (3.5), the normalized period matrix takes the form of (3.7). In this case, the LHF is given by

$$F_\infty^3 = \lim_{z \rightarrow 0} \exp\left(-\frac{\log z}{2\pi i}N\right) F_z^3 = \begin{bmatrix} 1 \\ 0 \\ f/2a \\ \pi \end{bmatrix}.$$

#### 4. GENERIC TORELLI THEOREM

The goal of this section is to show the generic Torelli theorem for one-parameter families of Calabi–Yau threefolds.

**4.1. Degree of period map.** Let  $\mathcal{X} \rightarrow S$  be a one-parameter family of Calabi–Yau threefolds. Given a smooth compactification  $\bar{S}$  of  $S$  so that  $\bar{S} - S$  consists of finite points. Let  $\phi : S \rightarrow \Gamma \backslash D$  be the period map associated to the VHS on  $H := H^3(X, \mathbb{Z})/\text{Tor}$  for a fixed smooth fiber  $X$ . Although the monodromy group  $\Gamma$  is not necessary a neat subgroup of  $G_{\mathbb{Z}}$ , there always exists a neat subgroup  $\Gamma'$  of  $\Gamma$  of finite index. In this situation, we have a lifting  $\tilde{\phi}$  of  $\phi$

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\phi}} & \Gamma' \backslash D \\ \downarrow & & \downarrow \\ S & \xrightarrow{\phi} & \Gamma \backslash D \end{array}$$

where  $\tilde{S}$  is a finite covering of  $S$ . To show the generic Torelli theorem for  $\phi$ , it suffices to show the theorem for the lifting  $\tilde{\phi} : \tilde{S} \rightarrow \Gamma' \backslash D$ . Therefore we henceforth assume that  $\Gamma$  is neat. We also assume that the Kodaira–Spencer map is an isomorphism on the base curve  $S$  to exclude trivial cases [BG]. To summarize, we assume that:

- (1) the monodromy group  $\Gamma$  is neat;
- (2) the Kodaira–Spencer map is an isomorphism on  $S$ .



Let  $\sigma_1, \dots, \sigma_n$  be the nilpotent cones which arise from the monodromies around  $\bar{S} - S$ . We define a fan  $\Xi$  in  $\mathfrak{g}_{\mathbb{R}}$  by

$$\Xi := \Gamma \cdot \sigma_1 \cup \dots \cup \Gamma \cdot \sigma_n \cup \{0\}.$$

The fan  $\Xi$  is strongly compatible with  $\Gamma$ . By [KU], the partial compactification  $\Gamma \backslash D_{\Xi}$  of  $\Gamma \backslash D$  is a logarithmic manifold and the period map extends to  $\phi : \bar{S} \rightarrow \Gamma \backslash D_{\Xi}$ . By Theorem 2.4, the image  $\phi(\bar{S})$  and the map  $\phi$  is analytic. Moreover  $\phi$  is proper and thus a finite covering map.

**Proposition 4.1.** *Let  $p \in \bar{S} - S$  be a MUM point. If  $\phi^{-1}(\phi(p)) = \{p\}$ , then the map  $\phi : \bar{S} \rightarrow \phi(\bar{S})$  is of degree 1.*

*Proof.* By Lemma 3.2, a disk  $\Delta_p$  around  $p$  is isomorphic to the image  $\phi(\Delta_p)$ . Since  $\phi(p)$  is not a branch point, the map  $\phi$  must be of degree 1.  $\square$

For  $p \in \bar{S} - S$ , the image  $\phi(p)$  is the nilpotent orbit determined by the local monodromy and the LHF around  $p$ . If the family has only one MUM, we clearly have  $\phi^{-1}(\phi(p)) = p$ , therefore the generic Torelli theorem holds by Proposition 4.1.

To show the generic Torelli theorem for a family with multi MUMs, it suffices to show that there exists a MUM point  $p_1$  such that for any other MUM point  $p_2$  the condition  $\phi(p_1) \neq \phi(p_2)$  holds. Let  $N_j$  be the logarithm of the local monodromy around  $p_j$ , and let  $F_j$  be the LHF. Then

$$\phi(p_j) = (\sigma_j, \exp(\sigma_{j,\mathbb{C}})F_j) \mod \Gamma$$

where  $\sigma_j = \mathbb{R}_{\geq 0}N_j$ . As discussed in the previous section, we have the normalized matrix (3.2) of  $N_j$  determined by  $a_j, b_j, e_j, f_j \in \mathbb{Q}$  and the canonical choice (3.7) of  $F_j$  determined by  $\pi_j \in \mathbb{C}$  using a symplectic basis  $e_3^j, \dots, e_0^j$  satisfying (3.1).

**Proposition 4.2.** *If  $b_1 \neq b_2$  or  $\pi_1 - \pi_2 \notin \mathbb{Q}$ , then*

$$g(\sigma_1, \exp(\sigma_{1,\mathbb{C}})F_1) \neq (\sigma_2, \exp(\sigma_{2,\mathbb{C}})F_2)$$

*for any  $g \in G_{\mathbb{Z}}$ . In other words, we have  $\phi(p_1) \neq \phi(p_2)$ .*

*Proof.* We define  $g \in G_{\mathbb{Z}}$  by  $e_k^1 \mapsto e_k^2$ . Then  $\text{Ad}(g)N_1$  is written as the normalized matrices determined by  $a_1, b_1, e_1, f_1$  using the symplectic basis  $e_3^2, \dots, e_0^2$ , and  $\text{Ad}(g)W(N_1) = W(N_2)$ . We put  $W = W(N_2)$ . If  $b_1 \neq b_2$ , there does not exist  $h \in G_{\mathbb{Z}}(W)$  such that  $\text{Ad}(hg)N_1 \in \sigma_2$  since  $b_1$  is invariant for the action of  $G_{\mathbb{Z}}(W)$  by Proposition 3.1. Then  $\text{Ad}(\gamma)\sigma_1 \neq \sigma_2 \mod \Gamma$  for any  $\gamma \in G_{\mathbb{Z}}$ .

Now suppose that  $b_1 = b_2$  and that there exists  $h \in G_{\mathbb{Z}}(W)$  such that  $\text{Ad}(hg)\sigma_1 = \sigma_2$ . The filtration  $gF_1$  is written as the normalized period matrix determined by  $\pi_1$  using  $e_3^2, \dots, e_0^2$ . Then the period matrix of the canonical choice in

$$hg \exp(\sigma_{1,\mathbb{C}})F_1 = \exp(\sigma_{2,\mathbb{C}})hgF_1$$

is determined by  $\pi_1 + \lambda$  with  $\lambda \in \mathbb{Q}$  since  $h \in G_{\mathbb{Z}}$  and  $N_2 \in \mathfrak{g}_{\mathbb{Q}}$ . Since  $\pi_1 - \pi_2 \notin \mathbb{Q}$ , we conclude that  $hg \exp(\sigma_{1,\mathbb{C}})F_1 \neq \exp(\sigma_{2,\mathbb{C}})F_2$ . Therefore there does not exist  $\gamma \in G_{\mathbb{Z}}$  such that  $\text{Ad}(\gamma)\sigma_1 = \sigma_2$  and  $\gamma \exp(\sigma_{1,\mathbb{C}})F_1 \neq \exp(\sigma_{2,\mathbb{C}})F_2$ .  $\square$

**Theorem 4.3** (Generic Torelli Theorem). *Let  $\mathcal{X} \rightarrow S$  be a one-parameter family of Calabi-Yau threefolds with a MUM point. Assume that there exists a MUM point  $p_1$  such that for any other MUM point  $p_2 \in \bar{S} - S$  the condition  $b_1 \neq b_2$  or  $\pi_1 - \pi_2 \notin \mathbb{Q}$  holds. Then the map  $\phi : \bar{S} \rightarrow \phi(\bar{S})$  is the normalization of  $\phi(\bar{S})$ .*

*Proof.* The assertion readily follows from the combination of Proposition 4.2 and Proposition 4.1.  $\square$

Theorem 4.3 in particular applies to the families of Calabi–Yau threefolds with exactly one MUM point. Such examples include almost all one-parameter mirror families of complete intersection Calabi–Yau threefold in weighted projective spaces and homogeneous spaces (see [vEvS] for more details). We will discuss some Calabi–Yau threefolds with two MUM points in the next section.

## 5. MIRROR SYMMETRY

In this section, we see that the Hodge theoretic invariants  $b$  and  $\pi$  appear in the framework of mirror symmetry. Mirror symmetry claims, given a family of Calabi–Yau threefolds  $\mathcal{X} \rightarrow B$  with a MUM point, there exists another family  $\mathcal{X}^\vee \rightarrow B^\vee$  of Calabi–Yau threefolds such that some Hodge theoretic invariants of  $X$  around the MUM point and symplectic invariants of  $X^\vee$  are equivalent in a certain way. Here  $X$  and  $X^\vee$  are generic members of  $\mathcal{X} \rightarrow B$  and  $\mathcal{X}^\vee \rightarrow B^\vee$  respectively. Simply put, mirror symmetry interchanges the complex geometry of one Calabi–Yau threefold  $X$  with the symplectic geometry of another, called a mirror threefold  $X^\vee$ , and such a correspondence depends on the choice of a MUM point. We should think that each MUM point corresponds to a mirror Calabi–Yau threefold. If a family of Calabi–Yau threefolds  $\mathcal{X} \rightarrow B$  has several MUM points, there should be several mirror threefolds. We refer the reader to [CK2] for a detailed treatment of mirror symmetry.

In this section, we investigate the interplay between the LMHS at a MUM point and the corresponding mirror threefold. For the sake of convenience, we restrict ourselves to one-parameter models, that is, the case when  $h^{2,1}(X) = h^{1,1}(X^\vee) = 1$ . Since the complex moduli space of  $X$  is 1-dimensional,  $X$  comes with a family  $\mathcal{X} \rightarrow S$  over a punctured curve  $S$ . Since mirror symmetry is a statement about a MUM point of  $S$ , we assume that such a point corresponding to  $X^\vee$  is chosen.

We denote by  $\Omega_z$  a holomorphic 3-form on the mirror Calabi–Yau threefold over a point  $z \in S$  of the family  $\mathcal{X} \rightarrow S$ . On an open disk  $\Delta$  around the MUM point  $z = 0$ , there exist solutions  $\omega_0, \dots, \omega_3$  of the Picard–Fuchs equation of the following form:

$$\begin{aligned}
 (5.1) \quad & \omega_3(z) = \psi_3(z) = 1 + O(z), \\
 & 2\pi i \omega_2(z) = \psi_3(z) \log(z) + \psi_2(z), \\
 & (2\pi i)^2 \omega_1(z) = 2\psi_2(z) \log(z) + \psi_3(z) \log(z)^2 + \psi_1(z), \\
 & (2\pi i)^3 \omega_0(z) = 3\psi_1(z) \log(z) + 3\psi_2(z) \log(z)^2 + \psi_3(z) \log(z)^3 + \psi_0(z),
 \end{aligned}$$

where  $\psi_i$  is a power series in  $z$  such that  $\psi_j(0) = 0$  for  $0 \leq j \leq 2$ . An important observation is that the local monodromy group at each MUM point is controlled by the topological invariants of the corresponding mirror threefold as follows. Let  $z_0 \in \Delta^*$  be a reference point. We equip  $H^3(X_{z_0}, \mathbb{Z})/\text{Tor}$  with the standard symplectic form (3.1). Then mirror symmetry predicts the existence of a symplectic basis  $A_0, A_1, B^1, B^0$  of

$H_3(X_{z_0}, \mathbb{Z})/\text{Tor}$  such that

$$(5.2) \quad \begin{bmatrix} \int_{A_0} \Omega_z \\ \int_{A_1} \Omega_z \\ \int_{B^1} \Omega_z \\ \int_{B^0} \Omega_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{c_2(X^\vee) \cdot H}{24} & \lambda & \frac{\deg X^\vee}{2} & 0 \\ \frac{\zeta(3)\chi(X^\vee)}{(2\pi i)^3} & -\frac{c_2(X^\vee) \cdot H}{24} & 0 & -\frac{\deg X^\vee}{6} \end{bmatrix} \begin{bmatrix} \omega_3(z) \\ \omega_2(z) \\ \omega_1(z) \\ \omega_0(z) \end{bmatrix},$$

where  $\lambda = 1$  if  $\deg X^\vee$  is even and  $= -1/2$  otherwise. This observation was first made in [CdOGP]. Although it is conjectural in general, it remains true for a large class of Calabi–Yau threefold, for example, those listed in [vEvS, Table 1].

**Proposition 5.1.** Assume the relation (5.2). Then the normalized matrix (3.2) of  $N = \log T$  and the normalized period matrix (3.7) of the LHF are determined by

$$a = 1, \quad b = \deg X^\vee, \quad e = \begin{cases} 1 & \text{if } b \text{ is even} \\ -1/2 & \text{if } b \text{ is odd,} \end{cases} \quad f = -\frac{c_2(X^\vee) \cdot H}{12}, \quad \pi = \frac{\chi(X^\vee)\zeta(3)}{(2\pi i)^3}.$$

*Proof.* The monodromy matrix of  $[\omega_3, \omega_2, \omega_1, \omega_0]^T$  is readily available. We rewrite it with respect to the symplectic basis to obtain  $N$ . The LHF is obtained in a similar manner.  $\square$

Therefore we see that the LMHS reflects the topological invariants of the mirror threefold. With this topological interpretation, the integrality condition (3.3) is explained by the Riemann–Roch theorem.

**Example 5.1.** For the mirror family of a quintic threefold  $X^\vee$ ,  $a, b, e, f$  and  $\pi$  are determined in [GGK, (III.A)]:

$$a = -1, \quad b = 5, \quad e = 11/2, \quad f = -25/6, \quad \pi = \frac{-200\zeta(3)}{(2\pi i)^3}.$$

Here  $\deg X^\vee = 5$ ,  $c_2(X^\vee) \cdot H = 50$  and  $\chi(X^\vee) = -200$ . By the base change (3.4), we change  $a$  and  $e$  into 1 and  $-1/2$  respectively.

**5.1. Multiple Mirror Symmetry.** We find Theorem 4.3 and Proposition 5.1 particularly interesting when a family of Calabi–Yau threefolds has two MUM points. Such a family is of considerable interest because the existence of two MUMs suggests the existence of two mirror partners. The first concrete example of such a multiple mirror phenomenon was discovered in [Rod]. Recently, a few more examples were constructed [Kan, HT, Miu]. In this section, we investigate two examples of Calabi–Yau threefolds with two MUM points.

**5.2. Grassmannian and Pfaffian Calabi–Yau threefold.** The Grassmannian  $\text{Gr}(2, 7)$  has a canonical polarization via the Plücker embedding into  $\mathbb{P}^{20}$ . The complete intersection of 7 hyperplanes sections of this embedding yields a Calabi–Yau threefold  $X^\vee := \text{Gr}(2, 7) \cap (1^7) \subset \mathbb{P}^{20}$  with  $h^{1,1} = 1$ .

Let  $N$  be a  $7 \times 7$  skew-symmetric matrix  $N = (n_{ij})$  with  $[n_{ij}]_{i < j} \in \mathbb{P}^{20}$ . The rank 4 locus determines a codimension 3 variety  $\text{Pfaff}(7)$  in  $\mathbb{P}^{20}$ , known as the Pfaffian. The complete intersection of 7 hyperplanes sections of the Pfaffian variety  $Y^\vee := \text{Pfaff}(7) \cap (1^{14}) \subset \mathbb{P}^{20}$  is a Calabi–Yau threefold with  $h^{1,1} = 1$ .

In each case, a mirror family is constructed by an orbifolding method [Rod]. An important observation is that the mirror families  $\mathcal{X} \rightarrow \mathbb{P}^1$  and  $\mathcal{Y} \rightarrow \mathbb{P}^1$  are identical and have exactly two MUM points: one corresponds to  $X^\vee$  and the other to  $Y^\vee$  (see Figure 1.). Mirror symmetry for this example was confirmed in [BCFKS, Tjo]. Since

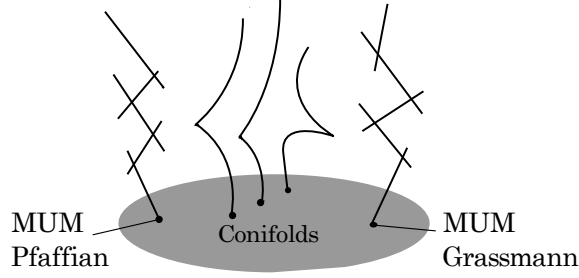


FIGURE 1. Moduli space  $\mathbb{P}^1$  of  $X = Y$

$\deg X^\vee = 42 \neq \deg Y^\vee = 14$ , Theorem 4.3 applies and the generic Torelli theorem holds for the mirror family. An identical argument applies to, for example, the Calabi–Yau threefolds constructed in [HT, Miu].

**5.3. Complete Intersection**  $\mathrm{Gr}(2, 5) \cap \mathrm{Gr}(2, 5) \subset \mathbb{P}^9$ . Let  $i_1, i_2 : \mathrm{Gr}(2, 5) \hookrightarrow \mathbb{P}^9$  be generic Plücker embeddings. It is shown in [Kan] that the complete intersection  $X^\vee := i_1(\mathrm{Gr}(2, 5)) \cap i_2(\mathrm{Gr}(2, 5))$  is a Calabi–Yau threefold with  $h^{1,1} = 1$ . In [Miu2], a mirror family  $\mathcal{X} \rightarrow \mathbb{P}^1$  of  $X^\vee$  was constructed via a toric degeneration of  $\mathrm{Gr}(2, 5)$  to a Hibi toric variety. An interesting observation is that the mirror family has exactly two MUM points and both of them correspond to  $X^\vee$ . The corresponding Hodge theoretic invariants around the MUM points are identical and Theorem 4.3 cannot be applied in this case. For the reader’s convenience, we write down the Picard–Fuchs operator with the Euler differential  $\Theta := z\partial_z$ :

$$\begin{aligned} & \Theta^4 - z(124\Theta^4 + 242\Theta^3 + 187\Theta^2 + 66\Theta + 9) \\ & + z^2(123\Theta^4 - 246\Theta^3 - 787\Theta^2 - 554\Theta - 124) \\ & + z^3(123\Theta^4 + 738\Theta^3 + 689\Theta^2 + 210\Theta + 12) \\ & - z^4(+124\Theta^4 + 254\Theta^3 + 205\Theta^2 + 78\Theta + 12) + z^5(\Theta + 1)^4 \end{aligned}$$

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MATHEMATICAL SCIENCE CENTER, TSINGHUA UNIVERSITY,  
 HAIDIAN DISTRICT, BEIJING 100084, CHINA.  
 tatsuki@math.tsinghua.edu.cn

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY  
 1 OXFORD STREET, CAMBRIDGE MA 02138 USA  
 kanazawa@10.alumni.u-tokyo.ac.jp